J87. Prove that for any acute triangle ABC, the following inequality holds:

$$\frac{1}{-a^2+b^2+c^2}+\frac{1}{a^2-b^2+c^2}+\frac{1}{a^2+b^2-c^2}\geq\frac{1}{2Rr}.$$

Proposed by Mircea Becheanu, Bucharest, Romania

First solution by Brian Bradie, VA, USA

Using the Law of Cosines and the formula

$$R=\frac{abc}{4rs},$$

we can rewrite the original inequality as

$$\frac{a}{\cos\alpha}+\frac{b}{\cos\beta}+\frac{c}{\cos\gamma}\geq 4s=2(a+b+c), \tag{1}$$

where α, β and γ are the acute angles in the triangle. Using the Law of Sines, we can write

$$c=a\frac{\sin\gamma}{\sin\alpha}\quad\text{and}\quad b=a\frac{\sin\beta}{\sin\alpha}.$$

Substituting into (1) yields

$$\tan\alpha+\tan\beta+\tan\gamma\geq 2(\sin\alpha+\sin\beta+\sin\gamma). \tag{2}$$

On $(0,\frac{\pi}{2})$, $\tan x$ is convex and $\sin x$ is concave; it therefore follows from Jensen's inequality that

$$\begin{aligned}\tan\alpha+\tan\beta+\tan\gamma &\geq 3\tan\left(\frac{\alpha+\beta+\gamma}{3}\right)=3\tan\frac{\pi}{3}=3\sqrt{3},\ \text{and}\\ \sin\alpha+\sin\beta+\sin\gamma &\leq 3\sin\left(\frac{\alpha+\beta+\gamma}{3}\right)=3\sin\frac{\pi}{3}=\frac{3\sqrt{3}}{2}.\end{aligned}$$

Hence, (2) holds with equality if and only if $\alpha=\beta=\gamma$. Thus, the original inequality holds with equality if and only if the triangle is an equilateral triangle.

Second solution by Mihai Miculita, Oradea, Romania

Because $2Rr=2\frac{S}{p}\cdot\frac{abc}{4S}=\frac{abc}{2p}=\frac{abc}{a+b+c}$, the given inequality is equivalent to

$$\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\geq\frac{abc}{a+b+c}. \quad (1)$$

Let us observe that since ABC is an acute triangle the following is true

$$\begin{aligned}
&b^2+c^2-a^2>0 \Rightarrow 2(b-c)^2(b^2+c^2-a^2) \geq 0 \\
&\Leftrightarrow (b-c)^2(2b^2+2c^2-2a^2) \geq 0 \\
&\Leftrightarrow (b-c)^2[(b+c)^2+(b-c)^2-2a^2] \geq 0 \\
&\Leftrightarrow (b^2-c^2)^2+(b-c)^4-2a^2(b-c)^2 \geq 0 \\
&\Leftrightarrow (b-c)^4-2a^2(b-c)^2+a^4 \geq a^4-(b^2-c^2)^2 \\
&\Leftrightarrow [a^2-(b-c)^2]^2 \geq (a^2+b^2-c^2)(a^2+c^2-b^2) \\
&\Leftrightarrow a^2-(b-c)^2 \geq \sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)} \\
&\Leftrightarrow (a+b-c)(a+c-b) \geq \sqrt{(a^2+b^2-c^2)(a^2+c^2-b^2)}. \quad (2)
\end{aligned}$$

Thus, using the AM-GM inequality and using the result in (2) we have that:

$$\begin{aligned}
\frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}\right) &\geq \frac{1}{\sqrt{(b^2+c^2-a^2)(a^2+c^2-b^2)}} \\
&\geq \frac{1}{(b+c-a)(a+c-b)}. \quad (3)
\end{aligned}$$

Summing up inequality (3) and the two obtained by a circular permutation of the letters we obtain

$$\begin{aligned}
\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2} &= \frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}\right) \\
&+\frac{1}{2}\left(\frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+b^2-c^2}\right)+\frac{1}{2}\left(\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2}\right) \\
&\geq \frac{1}{(b+c-a)(a+c-b)}+\frac{1}{(b+c-a)(a+b-c)} \\
&+\frac{1}{(a+c-b)(a+b-c)} \\
&= \frac{a+b+c}{(b+c-a)(a+c-b)(a+b-c)} \\
&\Rightarrow \frac{1}{b^2+c^2-a^2}+\frac{1}{a^2+c^2-b^2}+\frac{1}{a^2+b^2-c^2} \\
&\geq \frac{a+b+c}{(b+c-a)(a+c-b)(a+b-c)}. \quad (4)
\end{aligned}$$

It is known that

$$\sqrt{(b+c-a)(a+c-b)} \leq \frac{(b+c-a)+(a+c-b)}{2} = c.$$

Multiplying the above inequality with its respective ones obtained by circular permutation of letters we obtain

$$(b+c-a)(a+c-b)(a+b-c) \leq abc. \quad (5)$$

Using (4) and (5) we readily obtain the desired inequality (1).

Third solution by Ovidiu Furdui, Cluj, Romania

We will use the following standard trigonometric formulae

$$s = 4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \quad \text{and} \quad r = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2},$$

where s denotes the semiperimeter of triangle ABC. It is simply to check, by using the preceding formulas that $4sRr = abc$.

Let $f : (0, \frac{\pi}{2}) \to \mathbb{R}$ be the function defined by $f(x) = \frac{x}{\cos x}$. A calculation shows that

$$f''(x) = \frac{x + x\sin^2 x + \sin(2x)}{\cos^3 x} > 0,$$

and hence, f is a convex function. Using the Law of Cosines combined with Jensen's inequality for convex functions we get that

$$\frac{1}{-a^2+b^2+c^2} + \frac{1}{a^2-b^2+c^2} + \frac{1}{a^2+b^2-c^2} = \sum_{cyclic} \frac{1}{2bc\cos A} = \frac{1}{2abc}\sum_{cyclic}\frac{a}{\cos A}$$

$$\geq \frac{1}{2abc} \cdot 3 \cdot \frac{\frac{a+b+c}{3}}{\cos\frac{A+B+C}{3}} = \frac{2s}{abc} = \frac{1}{2Rr},$$

and the problem is solved.

Fourth solution by Tarik Adnan Moon, Kushtia, Bangladesh

$$\sum_{cyc} \frac{1}{-a^2+b^2+c^2} \geq \frac{1}{2Rr}$$

We know that, $-a^2 + b^2 + c^2 = 2bc \cdot \cos A$. So,we need to prove that,

$$\sum_{cyc} \frac{1}{2bc \cdot \cos A} \geq \frac{1}{2Rr}$$

Lemma 1: We know that, $[ABC] = sr = \frac{abc}{4R} \Longrightarrow 4sr = \frac{abc}{R}$

After multiplying by $2abc$ we get,

$$\sum_{cyc} \frac{a}{\cos A} \geq \frac{abc}{Rr} = \frac{4sr}{r} = 4s$$

By Cauchy-Schwarz inequaltiy we get,

$$\left(\sum_{cyc} a \cdot \cos A\right)\left(\sum_{cyc}\frac{a}{\cos A}\right) \geq \left(\sum_{cyc} a\right)^2 = 4s^2 ...(1)$$

Lemma 2: We know that,

$$\left(\sum_{cyc} a \cdot \cos A\right) = \frac{2sr}{R}$$

So, it is left to prove that,

$$\left(\sum_{cyc} a \cdot \cos A\right) = \frac{2sr}{R} \leq s \Leftrightarrow R \geq 2r$$

And we are done.

Some words about the lemmas:
Lemma 1: Straightforward, just need to use extended law of sines.
Lemma 2: We know that, $a\cos A = 2R\sin A \cdot \cos A = R \cdot \sin 2A$
Then,we use the identity, $\sum \sin 2A = 4\prod \sin A$
and using the extended law of sines we obtain, $4R^2 \prod \sin A = bc\sin A = 2[ABC]$
From these three we obtain, $\sum a\cos A = \frac{2[ABC]}{R} = \frac{2sr}{R}$

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